Journal of Contemporary Teacher Education
Vol: 8, 2024, 95-110

Impact of Calculus Teaching with a Blend of Traditional and Computational Techniques

Muhammad Atif Idrees*
Muhammad Arif Hussain**
Myong Ri***

Abstract

Challenges in learning mathematics can diminish students' interest in understanding mathematical concepts. With advancements in communication technology, a hybrid learning environment offers a promising solution to enhance the education system. Combining face-to-face classroom teaching with the use of application tools and systems proves beneficial in this context, fostering improved engagement and comprehension in education of Mathematics. Students frequently make mistakes when solving limit problems in calculus and are required to identify errors in attempts of their solutions. Even those proficient in performing specific procedures often have a limited understanding of the concepts and become confused when the problem context changes slightly. In addition, computer algebra systems (CAS) have been a topic of interest among educators for many years, eliciting both positive and negative feedback from researchers. This study used two main research methods: manual calculations and computer verification. First, different approaches were used to solve the problems by hand, and then Python, Maple, Mathematica, and Matlab software were used to check the results. By combining traditional problem-solving with advanced software, the study ensured that the answers were accurate and reliable. This approach was applied to solve both single-variable and multivariable limit problems typically found in undergraduate calculus courses and the result was good.

Keywords: Mathematics; computer algebra systems, Python, Maple, Mathematica, and Matlab; single variable and multivariable limits

---

* Government College Konkar Village Gadap Town, Karachi, Email: idreessmatif@gmail.com
** Karachi Institute of Economics and Technology, Karachi,
Email: arif.hussain@kiet.edu.pk
*** Faculty of Applied Mathematics, Kim Chaek University of Technology, DPR Korea, Email: rimyong.kut@gmail.com

Introduction

Students' interest in grasping mathematical concepts may wane when they encounter difficulties in their math classes. A hybrid learning environment is a viable way to improve the learning system thanks to communication technology improvements. In this context, it is advantageous to combine in-person classroom instruction with the usage of application tools and systems, as this promotes increased engagement and comprehension in mathematics education (Griffith, 2022). Calculus students often make mistakes when solving limit problems, and they must recognize these flaws when trying to solve the issues. Even experts in executing particular techniques frequently possess a restricted comprehension of the ideas and get perplexed when the issue setting significantly shifts (Abd Halim et al., 2019). Moreover, researchers have expressed both favorable and unfavorable opinions about Computer Algebra Systems (CAS), a subject of interest for educators for a considerable amount of time (Carvalho, Xu, Zotos & Kado, 2022 & 2023).

With technological advancements and the emergence of new challenges that previous generations did not encounter, there is a heightened need for individuals who appreciate mathematics, possess advanced mathematical thinking, and can apply mathematics in modeling and problem-solving. These skills are gradually necessary in addressing the complex glitches of today's world (Daniel, 2018). Computing has come to be a critical resource in modern research, permeating a diverse array of technical disciplines. Advances in symbolic algebra tools have proven especially beneficial in the mathematical sciences, where complex calculations in fields such as calculus and theoretical physics extend beyond the capabilities of manual computation. These developments have facilitated more efficient and practical approaches to solving intricate mathematical problems, thereby advancing research and discovery (Hill, 20220). Computer algebra, also known as symbolic computation, emerged as a significant field in the late 20th century, contributing to the development of computer algebra systems. These systems have revolutionized the manipulation and analysis of mathematical objects, particularly polynomials, by enabling sophisticated algorithmic processes. The advancements in CAS have markedly accelerated computational tasks, enhancing both the efficiency and scope of mathematical research and applications (Earl, 2017). Additionally, robust graphical capabilities and the ability to perform both symbolic and numeric mathematical computations are essential components of an ideal scientific computing environment (Shi, 2007). Moreover, this paper investigates how a deliberate combination of symbolic and numeric computational tools can lead to improved problem-solving abilities in a particular topic. The main objective is to use symbolic computation to automatically generate stable and effective numerical calculation programs (Drijvers, 2000). The learning of calculus has usually posed noteworthy challenges within higher education, often unapproachable students. The integration of symbolic computation software with calculus instruction offers intuitive, visual aids that enhance student comprehension. This approach simplifies complex concepts, fosters interest in learning, and improves academic performance. However, it is crucial to mitigate the risk of students developing an over-reliance on such software, which can lead to poor learning habits. Educators must strive to balance the use of these tools, ensuring that students develop a proper understanding and appreciation of concept of Calculus and the Computational aids available (Zsolt, 2024).

The specific focus of this paper is a few limit problems found in calculus. Hybrid symbolic-numeric algorithms have significant potential across various problem areas. Start with the example of solution of differential equations and the development of programs for the efficient numerical estimation of functions. Moreover, the initial experiences with this computational mode have been extremely positive, demonstrating its potential to improve mathematical problem-solving skills. We simplify the integration problem to an interval with a left endpoint at zero and no singularities within the interval, maybe only at zero, using the methods covered in this paper. If singularities are found outside of the interval, we seek the singularity closest to zero and use methods for treating singularities to try and remove it. When the left endpoint is a singular point, singularity-handling techniques are directly applied to that point. The fundamental tool for managing singularities is the generalized series expansion (Hussain et al., 2014). Despite the extensive literature on the power of Computer Algebra Systems (CAS), their commands, and the programming aspects, there is a noticeable lack of discussion regarding the limitations, inadequacies, and potential for misuse of these systems (Baumbach et al., 2004). This gap in the literature often leads beginning undergraduate students to mistakenly assume that the "black box" software can solve any Mathematics related problem accurately (Apostol, 1974).

This paper aims to highlight some of the shortcomings of a few of the most popular CASs, namely Maple, Python, Matlab and Mathematica. We present examples from typical calculus courses where some CASs produce incomplete, inaccurate, or misleading results. Each section begins with examples where CASs produced inaccurate results, which were later corrected in more recent versions. The sections then conclude with examples and actual output from CASs where the software still struggles to produce accurate results. Although we have many examples at our disposal, we have limited our presentation to those that best demonstrate the software's shortcomings. In below, we discuss solving limits problems using different methods. Next is devoted to limits problem solutions from CASs, and deals with comparison of outputs from these applications. In depth results which provide a comprehensive understanding of the subject, are discussed in last. Many examples are based on the authors' classroom experiences.

Solving limits problems using different methods

In this study, we focus on two types of limit problems, providing examples of two single variable and one multivariable limit problems and utilizing different approaches to determine their exact solutions (Royden & Fitzpatrick, 2009, 2010). Additionally, we employ Python, Maple, Matlab and Mathematica tools to calculate the approximations of these limits and their closed forms, thereby verifying our solutions. This dual approach ensures both the accuracy of the theoretical skills and in hand utility of computational tools in Calculus.

Example 1

$$\underset{x\to 3}{Lim}\frac{x!-6}{x-3}$$

<u>**Method I**</u> L' Hospital Rule

$$\underset{x\to 3}{Lim}\frac{x!-6}{x-3} = \lim_{x\to 3}\frac{\frac{d}{dx}(x!-6)}{\frac{d}{dx}(x-3)} = \lim_{x\to 3}\frac{d}{dx}(x!) \qquad (1)$$

Now, consider

Gamma function $x! = \Gamma(x+1) = x\,\Gamma(x) = x(x-1)(x-2)\,\Gamma(x-2) = (x^3 - 3x^2+2x)\,\Gamma(x-2)$

$\frac{d}{dx}(x!) = (3x^2 - 6x + 2)\,\Gamma(x-2) + (x^3 - 3x^2 + 2x)\,\Gamma^{/}(x-2)$

Now, applying limit, we get

$\lim_{x\to 3}(\frac{d}{dx}(x!)) = (27 - 18 + 2)\,\Gamma(1) + (27 - 27 + 6)\,\Gamma^{/}(1)$

$= 11\Gamma(1) + 6\Gamma^{/}(1) = 11 + 6(-\gamma)$ where $\Gamma^{/}(1) = \int_0^{\infty} e^{-x} ln x\, dx = -\gamma =$ Euler's constant

And $\Gamma(1) = 0! = 1$, $\gamma \approx 0.57721$ to 5 decimal places.

$=11 - 6(\gamma)$, this is the exact answer. The approximate answer is given below.

$= 11 - 6(0.57721) = 7.53674$

Now, applying method 2.

Method II Using Stirling Formula:

By using Stirling formula $\mathbf{x!} \approx \sqrt{2\pi x}\ (x/e)^x$

$\frac{d}{dx}(x!)$ = x! $(\frac{1}{2x} + lnx)$

$\therefore \lim_{x\to3}(x!)$ = 3! ($\frac{1}{6} + ln3$) = 6($\frac{1}{6}$ + ln3) = 1 + 6ln3 = 1 + 6(1.098) = 7.588 to 3 decimal places.

This tallies the answer as earlier. Now, we consider third method.

Method III Taylor's Expansion

By property of Gamma function

f(x) = x! – 6 = x(x−1) (x−2)$\Gamma(x-2) - 6$

$\Longrightarrow f^{/}$(x) = (x^3 – 3x^2 + 2x) $\Gamma^{/}$(x − 2) + (3x^2 – 6x + 2)$\Gamma(x-2)$

$\Longrightarrow f^{//}$(x) = (x^3 – 3x^2 + 2x) $\Gamma^{//}$(x − 2) + 2(3x^2 – 6x + 2)$\Gamma^{/}$(x − 2) + 6(x – 1)$\Gamma(x-2)$

Similarly *n*th order derivatives

f(3) = 3! – 6 = 6 – 6 = 0

$f^{/}$(3) = (27 – 27 + 6)$\Gamma^{/}$(1) + (27 – 18 + 2)$\Gamma(1)$ = 6 $\Gamma^{/}$(1) + 11(0!) = 11 – 6 γ

$f^{//}$(3) = 6$\Gamma^{//}$(1) + 2(11)$\Gamma^{/}$(1) + 12Γ(1) = 6($\frac{\pi^2}{6}$ + γ^2) + 22(– γ) + 12(1) = π^2 + 2(3γ – 2)(γ – 3)

Derivatives are curtailed at this stage due to diminishing contributions in proceeding limit and complexity acquiring Gamma and Digamma functions.

Taylor's expansion at a = 3: f(x) = $\sum_{n=0}^{\infty}(x-3)^n \frac{f^{(n)}(3)}{n!}$

$\Longrightarrow f$(x) = 0 + (x – 3)$\left\{f^{/}(3) + \sum_{n=2}^{\infty}\frac{(x-3)^{(n-1)}}{n!}\ f^{(n)}(3)\right\}$, Again, we get the same answer.

$\therefore \lim_{x\to3}\frac{x!-6}{x-3} = \lim_{x\to3}\left\{f^{/}(3) + \sum_{n=2}^{\infty}\frac{(x-3)^{(n-1)}}{n!}\ f^{(n)}(3)\right\}$ = $f^{/}$(3) = 11 – 6 γ = 7.53674

Next, we consider example 2.

Example 2

$$\underset{n\to\infty}{Lim}\frac{(n!)^{\frac{1}{n}}}{\mathrm{n}}$$

Solving above example by Sterling formula collaborating with L' Hospital rule:

Let L = $\underset{n\to\infty}{Lim}\frac{(n!)^{\frac{1}{n}}}{\mathrm{n}} = \lim_{n\to\infty}\frac{1}{n}\left(\sqrt{2\pi n}(\frac{n}{e})^n\right)^{\frac{1}{n}} = \frac{1}{e}\lim_{n\to\infty}(2\pi n)^{\frac{1}{2n}}$

$\Longrightarrow \ln L = \lim_{n\to\infty}\{ln\frac{1}{e} + \frac{1}{2n} ln(2\pi n)\} = \lim_{n\to\infty}\{-1 + \frac{ln(2\pi n)}{2n}\} = -1$
$+ \lim_{n\to\infty} \frac{ln(2\pi n)}{2n}$ $(\frac{\infty}{\infty}\ form)$

$= -1 + \lim_{n\to\infty} \frac{\frac{1}{2\pi n}2\pi}{2} = -1 + \lim_{n\to\infty} \frac{1}{2n} = -1$

$\Longrightarrow$ L = $\underset{n\to\infty}{Lim} \frac{(n!)^{\frac{1}{n}}}{\text{n}} = \frac{1}{e} = e^{-1}$

Now, consider method 2.

<u>Method II</u> By Squeeze Theorem inculcating Rieman's sum for increasing function,
furnishing with L' Hospital rule

*ln*x is increasing function over [0, n], so have the inequalities:

(1) $\int_1^n lnx\ dx = \int_1^2 lnx\ dx + \int_2^3 lnx\ dx + \ldots\ldots.. + \int_{n-1}^n lnx\ dx$

$< \int_1^2 \ln(2)\ dx + \int_2^3 \ln(3)\ dx + \ldots\ldots.. + \int_{n-1}^n \ln(n)\ dx$

Replacing *f* at upper end points in each integral

$=$ ln(2) + ln(3) + ……….+ ln(n) = $\ln(2\cdot 3 \cdot \ldots \ldots \cdot n) =$ ln(n!)

That is $\int_1^n lnx\ dx <$ ln(n!) ----------------[1]

And (2) $\int_1^{n+1} lnx\ dx = \int_1^2 lnx\ dx + \int_2^3 lnx\ dx + \ldots\ldots.. + \int_n^{n+1} lnx\ dx$

$> \int_1^2 \ln(1)\ dx + \int_2^3 \ln(2)\ dx + \ldots\ldots.. + \int_n^{n+1} \ln(n)\ dx$

Replacing *f* at lower end points in each integral

= ln(1) + ln(2) + …….. + ln(n) = $\ln(1\cdot 2\cdot \ldots \ldots.. \cdot n)$ = ln(n!)

That is $\int_1^{n+1} lnx\ dx >$ ln(n!) ----------------[2]

From [1] and [2] $\int_1^{n+1} lnx\ dx <$ ln(n!) $< \int_1^n lnx\ dx$

$\Longrightarrow$ n *ln* n – (n-1) < ln(n!) < (n + 1) ln(n + 1) – n

LHS & RHS are integrated by parts

$\Longrightarrow$ $\ln\{n^n e^{-(n-1)}\} < \ln(n!) < \ln\{(n+1)^{n+1}e^{-n}\}$
$\Longrightarrow$ $n^n e^{-(n-1)} < (n!) < (n+1)^{n+1}e^{-n}$
$\Longrightarrow$ $\left(\frac{n}{e}\right)^n e < (n!) < (\frac{n+1}{e})^n(n+1)$
$\Longrightarrow$ $\frac{n}{e}e^{\frac{1}{n}} < (n!)^{\frac{1}{n}} < \left(\frac{n+1}{e}\right)(n+1)^{\frac{1}{n}}$
$\Longrightarrow$ $\frac{1}{e}e^{\frac{1}{n}} < \frac{(n!)^{\frac{1}{n}}}{n} < \frac{1+\frac{1}{n}}{e}(n+1)^{\frac{1}{n}}$
$\Longrightarrow$ $\frac{1}{e}\lim_{n\to\infty} e^{\frac{1}{n}} < \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} < \lim_{n\to\infty}\frac{1+\frac{1}{n}}{e}(n+1)^{\frac{1}{n}}$
$\Longrightarrow$ $\frac{1}{e}\lim_{n\to\infty} e^{0} < \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} < \frac{1}{e}\lim_{n\to\infty}(n+1)^{\frac{1}{n}}$ ----------------[3]

Consider

$$l = \lim_{n\to\infty}(n+1)^{\frac{1}{n}} \qquad (\infty^0\ form)$$

$\Longrightarrow ln(l) = \lim_{n\to\infty} ln(n+1)^{\frac{1}{n}} = \lim_{n\to\infty}\frac{1}{n}ln(n+1) = \lim_{n\to\infty}\frac{\ln(n+1)}{n}$
($\frac{\infty}{\infty}$ form)

$$= \lim_{n\to\infty}\frac{\frac{1}{n+1}}{1} = \lim_{n\to\infty}\frac{1}{n+1} = 0 \qquad \Longrightarrow l = 1 \text{----------------[4]}$$

From [3] and [4] $\frac{1}{e} < \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} < \frac{1}{e}$

$\therefore \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$ By Squeeze Theorem

Next, we consider method 3.

Method III Using Ratio & Root Tests in consideration of testing series for convergence

Fact to use here: If $a_n > 0$ and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}$, $\lim_{n\to\infty}\sqrt[n]{a_n}$ both exist and are finite then they are equal.

We know that if

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = L \qquad ********(1)$$

then for any $\varepsilon > 0$ there is N$\in \mathbb{N}$(set of natural numbers) such that

$$L - \varepsilon < \frac{a_{n+1}}{a_n} < L + \varepsilon \qquad \forall\ n \geq N$$

Scale this down all the way from n up to N. Hence taking their product, we obtained

$$(L - \varepsilon)^{n+1-N} < \frac{a_{n+1}}{a_N} < (L + \varepsilon)^{n+1-N}$$

Isolating a_{n+1} and rearranging LHS & RHS we have

$$(L - \varepsilon)^{n+1} \frac{a_N}{(L-\varepsilon)^N} < a_{n+1} < (L+\varepsilon)^{n+1} \frac{a_N}{(L+\varepsilon)^N}$$

Taking (n+1)th root of all part of inequality :

$$(L - \varepsilon) \sqrt[n+1]{\frac{a_N}{(L-\varepsilon)^N}} < \sqrt[n+1]{a_{n+1}} < (L + \varepsilon) \sqrt[n+1]{\frac{a_N}{(L+\varepsilon)^N}}$$

$$(L - \varepsilon)\left(\lim_{n\to\infty} \sqrt[n+1]{\frac{a_N}{(L-\varepsilon)^N}} \right) < \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} \right) < (L + \varepsilon)\left(\lim_{n\to\infty} \sqrt[n+1]{\frac{a_N}{(L+\varepsilon)^N}} \right)$$

Limits attached to ($L\mp\varepsilon$) tend to 1 as $n\to \infty$

$\Rightarrow$ $(L - \varepsilon)(1) < \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} \right) < (L + \varepsilon)(1)$

$\Rightarrow$ $(L - \varepsilon) < \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} \right) < (L + \varepsilon)$

$\Rightarrow$ $\lim_{n\to\infty} \sqrt[n]{a_n} = L$

(2)

[n+1 is replaced by n without loss of generality]

Hence from (1) and (2) $\lim_{n\to\infty} \sqrt[n]{a_n} = L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

(3)

Next write the expression as *n*th root

$$\frac{(n!)^{\frac{1}{n}}}{n} = \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$$

Let $a_n = \frac{n!}{n^n}$ $\Rightarrow$ $a_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$

$\therefore$ $\frac{a_{n+1}}{a_n} = \left(\frac{n}{n+1}\right)^n = \left(\frac{n+1}{n}\right)^{-n}$

$\Longrightarrow \lim_{n\longrightarrow\infty} \frac{a_{n+1}}{a_n} = \lim_{n\longrightarrow\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\longrightarrow\infty} \left(\frac{n+1}{n}\right)^{-n} = e^{-1} = \frac{1}{e}$ ********(4)

Finally (3) and (4) imply

$$\lim_{n\longrightarrow\infty} \frac{(n!)^{\frac{1}{n}}}{n} = \frac{1}{e}$$

Now, consider method 4.

<u>Method IV</u> Using Rieman's sum as area under the curve

Writing the limit as $L = \underset{n\to\infty}{Lim} \frac{(n!)^{\frac{1}{n}}}{n} = \lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

The limit is now in 0^0 form, so that

$$lnL = \lim_{n\to\infty} \frac{1}{n} ln\left(\frac{n!}{n^n}\right) = \lim_{n\to\infty} \frac{1}{n} ln\left[\frac{n(n-1)\ldots\ldots 3\cdot 2\cdot 1}{n\cdot n \ldots\ldots\ldots n\cdot n\cdot n}\right]$$

$$= \lim_{n\to\infty} \frac{1}{n} \ln\left[\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\ldots\ldots\left(\frac{n-1}{n}\right)\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\ln\left(\frac{1}{n}\right) + \ln\left(\frac{2}{n}\right) \ldots.. + \ln\left(\frac{n-1}{n}\right) + \ln\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n}\ln\left(\frac{1}{n}\right) + \frac{1}{n}\ln\left(\frac{2}{n}\right) \ldots.. + \frac{1}{n}\ln\left(\frac{n-1}{n}\right) + \frac{1}{n}\ln\left(\frac{n}{n}\right)\right]$$

$= \lim\limits_{n\to\infty}$ {Sum of rectangles of height at right hand end points over common simplified interval

[0,1] with regular partition of width $\triangle x = \frac{1}{n}$ of function f(x) = lnx}

$$= \int_0^1 lnx\,dx = \lim_{t\to 0}\int_t^1 lnx\,dx = \lim_{t\to 0}\,[xlnx - x]_t^1$$

$$= \lim_{t\to 0}\{(1\cdot ln1 - 1) - (tlnt - t)\} = 0 - 1 - \lim_{t\to 0}(tlnt)$$

$[0\times\infty\ limit\ form]$

$$= -1 - \lim_{t\to 0}\frac{lnt}{\frac{1}{t}} = -1 - \lim_{t\to 0}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = -1 + \lim_{t\to 0}t = -1 + 0 = -1$$

$[\frac{\infty}{\infty}\ limit\ form]$

$$\therefore\ ln\text{L} = -1 \qquad \Longrightarrow \qquad \text{L} = \lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{\text{n}} = e^{-1} = \frac{1}{e}$$

In all cases, we get the same answer.
Now, we consider example 3 for Multivariable Limit case.

Example 3

Path Dependency in Multivariable Limits.

Find the Limit: $\lim\limits_{(x,y)\to(0,0)}\frac{yx^2}{x^2+y^2}$

To evaluate this limit, we find it along different paths.

1. Along x-axis (where y = 0):

$$\frac{yx^2}{x^2+y^2} = \frac{0.x^2}{x^2+0} = 0$$

Therefore, along x-axis, the limit is 0.

2. Along y-axis (where x = 0):

$$\frac{y.0}{0+y^2} = 0$$

Therefore, along y-axis, the limit is 0.

3. Along the line y = kx, for some constant k:

$$\frac{kxx^2}{x^2+x^2k^2} = \frac{kx}{k^2+1} = 0$$

As (x,y) → (0,0), x→0, therefore, limit →0.

4. Using polar coordinates:
   Let x = r cosθ, and y = r sinθ. Then

$$\frac{yx^2}{x^2+y^2} = \frac{r^3 Sin\theta\ Cos^2\theta}{r^2} = r\ \text{Sin}\theta\ \text{Cos}^2\theta$$

As r→0, above expression →0. Therefore, limit →0.
Since the limit is 0 along different paths approaching (0, 0), we conclude that
limit of $\frac{yx^2}{x^2+y^2}$ is 0.

Next, we solve these examples using Computer Algebra Systems.

Solving limits problems using different Computer Algebra System

In this work, we employed Python, Maple, Matlab, and Mathematica tools to calculate the limits of various mathematical functions [20-23]. Python, particularly through the use of NumPy, generated accurate numeric solutions to the limit problems, demonstrating their reliability and precision in numerical computation. While SymPy, a Python library for symbolic mathematics, is robust and versatile, our examples highlight scenarios where advanced mathematical techniques and deeper analytical approaches are necessary. These cases extend beyond the current capabilities of SymPy's built-in functionalities, leading to incorrect exact solutions for multivariable limits in some cases.

In contrast, NumPy, which focuses on numerical operations, produced accurate numeric solutions for these examples. This underscores the significance of selecting the suitable tool based on the nature of the problem at hand. Matlab in some cases produce incorrect numerical results if **syms** command is not used properly. While symbolic computation is invaluable for exact solutions and theoretical analysis, numeric approaches can offer practical and precise results when symbolic methods fall short. Computer Algebra Systems (CAS) are dominant tools for solving an extensive range of mathematical problems, offering both symbolic and numeric computation skills. However, these systems can encounter complications when dealing with certain types of multivariable limit problems.

Maple, and Mathematica produced correct exact solutions to these limit problems. Our findings emphasize the complementary roles of different computational tools in mathematical problem-solving. By leveraging the strengths of each system, we can address a broader spectrum of mathematical challenges effectively.

Comparison of Solutions using different Computer Algebra System
In this section we give details of computer commands used to produce the limits solutions of three examples. Tables 1 and 2 contain the inputs and outputs of commands.

Table 1

Showing codes from Python and Matlab and their outputs

Examples	Code from Python	Code from Matlab

1	import sympy as sp
	# Define the variable
	x = sp.Symbol('x')
	# Define the factorial function
	factorial = sp.factorial(x)
	# Define the function f(x)
	f = (factorial - 6) / (x - 3)
	# Compute the limit as x approaches 3
	limit_value = sp.limit(f, x, 3)
	print(limit_value)
	Output = 11 - 6*Eulergamma
		% Define the symbolic variable
		syms x;
		% Define the function
		f = (factorial(x) - 6) / (x - 3);
		
		% Compute the limit as x approaches 3
		limit_value = limit(f, x, 3);
		
		% Display the result
		disp(limit_value);
		Output = 11 - 6*Eulergamma

2	import sympy as sp
	# Define the variable
	n = sp.Symbol('n', integer=True)
	# Define the factorial function and the expression
	factorial_n = sp.factorial(n)
	expr = factorial_n / n**n
	# Define the expression to be raised to the power of 1/n
	expr_power = expr**(1/n)
	# Compute the limit as n approaches infinity
	limit_value = sp.limit(expr_power, n, sp.oo)
	print(limit_value)
	Output = 1/e
		% Define the symbolic variable
		syms n;
		
		% Define the expression
		expr = (factorial(n)) / (n^n);
		
		% Compute the limit of the nth root of the expression as n approaches infinity
		limit_value = limit(expr^(1/n), n, Inf);
		% Display the result
		disp(limit_value);
		Output = 1/e

3	import numpy as np
	# Define a small delta to get close to the origin
	delta = 1e-8
	# Create a grid of points close to (0, 0)
	x_vals = np.linspace(-delta, delta, 100)
	y_vals = np.linspace(-delta, delta, 100)
	x, y = np.meshgrid(x_vals, y_vals)
	# Evaluate the function (yx^2)/(x^2 + y^2) at these points
	z = (y * x**2) / (x**2 + y**2)
	# Calculate the mean value to approximate the limit
	approx_limit = np.mean(z)
	approx_limit
	Output: 0.0
		syms x y m;
		f = (y * x^2) / (x^2 + y^2);
		
		% Check limit along the line y = mx
		f_m = subs(f, y, m*x);
		
		% Compute the limit as x approaches 0
		limit_value = limit(f_m, x, 0);
		disp(limit_value);
		
		Output: 0

Table 2

showing codes from Maple and Mathematica and their ouputs.

Examples	Code from Maple	Code from Mathematica

1	f := (x) -> (x! - 6) / (x - 3);
	limit(f(x), x = 3);
	
	Output = $11 - 6\gamma$
		f[x_] := (Factorial[x] - 6) / (x - 3);
		Limit[f[x], x -> 3]
		
		Output = $11 - 6\gamma$

2	expr := (n!) / (n^n);
	limit(expr^(1/n), n = infinity);
	
	Output = 1/e
		expr = (n!)/(n^n);
		Limit[expr^(1/n), n -> Infinity]
		
		Output = 1/e

3	f := (x, y) -> (y*x^2)/(x^2 + y^2);
	limit(f(x, y), {x = 0, y = 0});
	
	Output: 0
		(* Define the function *)
		f[x_, y_] := (y * x^2) / (x^2 + y^2)
		(* Compute the limit as (x, y) -> (0, 0) *)
		Limit[f[x, y], {x, y} -> {0, 0}]
		Output: 0

Results and Discussions

The specific focus of this paper is on a few limit problems encountered in calculus, highlighting the potential of hybrid symbolic-numeric algorithms across various problem areas. Two of the examples discussed are single-variable limit problems, while the third example involves a multivariable limit problem, necessitating the use of different paths to evaluate the limit accurately. The results of our experiments with this computing method have been really encouraging, underscoring its potential to advance mathematical problem-solving capabilities. By employing the techniques discussed in this paper, we simplify the limit problem to arrive at solutions using various methods available in the literature. A computer algebra system is a vital and intelligent tool that serves as a comprehensive math toolbox for engineers and other professionals dealing with complex and time-consuming tasks. Its wide array of features, including powerful computational capabilities and advanced visualization tools, make it particularly beneficial in the instructional process. The ability to visually represent mathematical concepts and solutions is highly appealing to users, enhancing understanding and engagement. Moreover, CAS tools are well-suited to learner-centered teaching methods, allowing for interactive and dynamic learning experiences that cater to the needs of students. Overall, the integration of CAS in educational settings supports a deeper and more intuitive grasp of mathematical and engineering concepts. That is why, we then utilized computer algebra systems (CAS) to obtain results for these limit problems. We have identified handling techniques to mitigate any ambiguity, emphasizing the importance of selecting the appropriate libraries available in different computer applications. This is clear from Tables 1 and 2.

Despite the extensive literature on the power of Computer Algebra Systems (CAS), including their commands and programming aspects, there is a noticeable lack of discussion regarding their limitations, inadequacies, and potential for misuse. This gap in the literature often leads beginning undergraduate students to mistakenly assume that these “black box” software solutions can accurately solve any mathematical problem without fail.

Conclusion

This paper provided the theoretical solutions in various ways and compared them with the outputs from these CASs, highlighting specific instances where the software struggled to deliver accurate results. Our discussion highlights the significance of understanding the capabilities and constraints of CASs, as these tools, while powerful, are not infallible. By shedding light on these limitations, we aim to assist both students and educators in making more informed decisions when using these tools in academic settings and as AI technologies continue to advance, they will be integrated into instruction in the future, providing innovative tools that can improve student engagement and understanding of complex mathematical concepts.

References

Griffith, D. A. (2022). Computer-Algebra-Software-Assisted Calculus instruction, not calculus for dummies: Bespoke Applications Necessitate Theory. *Applied Math, 2*(2), 261-268.

Abd Halim, S., Alwadood, Z., Noor, N. M., & Sulaiman, H. (2019). Development of quadric surfaces learning tool for engineering students using MATLAB GUI. *Journal of Advanced Research in Computing and Applications, 17*(1), 6-13.

Carvalho, P., Descalço, L., & Gonçalves, H. F. (2023). Using computer algebra systems in teaching and assessment in calculus. *In EDULEARN23 Proceedings* (pp. 2041-2048). IATED.

Xu, L., & Chen, K. (2022). Application research of mathematica software in calculus teaching. *Applied Mathematics and Nonlinear Sciences, 8*(1), 1785-1792.

Zotos, K., & Atanassova, I. (2023). Advantages & disadvantages of online CAS. *Journal of Software Engineering and Applications, 16*(9), 496-504.

Kado, K. (2022). A teaching and learning the fundamental of calculus through Python- based coding. *International Journal of Didactical Studies, 3*(1), 15006. https://doi.org/10.33902/IJODS.202215006

Daniel, V. (2018). “Physics simulations in python”. Unpublished Work. url: https://physics.weber.edu/schroeder/scicomp/PythonManual.pdf

Hill, C. (2020). Learning scientific programming with Python. *Cambridge University Press.*

Earl, R. (2017). Towards higher mathematics: A companion. *Cambridge University Press.*

Shi, M., & Bichelmeyer, B. A. (2007). Teachers' experiences with computers: A comparative study. *Journal of Educational Technology & Society, 10*(2), 180-190.

Drijvers, P. (2000). Students encountering obstacles using a CAS. *International Journal of Computers for Mathematical Learning*, *5*, 189-209..

Zsolt, L. (2024). “The use of Computer Algebra Systems” Internet: http://tsg.icme11.org/document/get/238 [July.5,]

Hussain, M. A., Khan, M. A., Shaheen Abbas, M. I., Jan, B., & Ansari, M. R. K. (2014). The impact of technology in mathematics education: a case study of Pakistan. *Journal of Basic & Applied Sciences*, *10*, 67-72..

Baumbach, D., Christopher, T., Fasimpaur, K., & Oliver, K. (2004). Personal assistants literacy using handhelds for literacy instruction. *Learning & Leading with Technology*, *32*(2), 16-21.

Apostol, T. M. (1974). *Mathematical Analysis* (2nd ed.). Addison-Wesley.

Royden, H. L., & Fitzpatrick, P. M. (2010). *Real Analysis* (4th ed.). Pearson.

Fitzpatrick, P. M. (2009). *Advanced Calculus* (2nd ed.). American Mathematical Society.

Stewart, J. (2011). *Multivariable Calculus* (7th ed.). Cengage Learning

Python Software Foundation. (2023). Python 3.11. Official Python Documentation. Retrieved from https://docs.python.org/3.11/ (accessed on July 30, 2024)

Maplesoft. (2023). Maple User Manual. Retrieved from https://www.maplesoft.com/documentation_center/(accessed on July 30, 2024)

MathWorks. (2023). MATLAB R2023b Documentation. Retrieved from https://www.mathworks.com/help/matlab/(accessed on July 30, 2024)

Wolfram Research. (2023). Mathematica Documentation Center. Retrieved from https://reference.wolfram.com/language/ (accessed on July 30, 2024)

Citation of this Article: Idrees, M. A., Hussain, M. A., & Ri, M. (2024). Impact of calculus teaching with a blend of traditional and computational techniques. *Journal of Contemporary Teacher Education, 8*, 95-110.